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Gunter's Line - Standard Celeration Chart Ancestor, Not Napier's Logs or Bones

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Historian Eshleman

John Eshleman's interest and production position him as the historian of Standard Celeration Charting. His bibliography of 2,668 printed and presented Precision Teaching references is masterful (Eshleman 1983, 1990a, 1990b).

Now, John reminds us to celebrate the fact that 400 years ago, in 1594, John Napier developed his wonderful table of logarithms. Eshleman also shows us how to use a table of logarithms. Because inexpensive pocket calculators are everywhere, neither logarithms nor slide rules have been used recently. Hence, neither is understood.

I am told that logarithms are still taught in many high schools to solve equations like $7^x = 24$, and word problems like "How many years will it take $1,000 to double at 6% compounded annually?" (Auman, 1994).

Although the logarithm is related to the Standard Celeration Chart (SCC), it is a mistake to think of it as a direct ancestor. The direct Standard Celeration Chart ancestor is Gunter's multiply line. Logarithms are arithmetic, algebraic, and digital. Gunter's line is geometric and analogical. We read indirectly numbers from logarithmic tables. We see directly multiply distances on Gunter's line and on our Standard Celeration Chart.

Multiplication inventions from 1594 through 1622

A flurry of inventions for simplifying the difficult and cumbersome multiplication, division, roots and powers centered in the British Isles from 1594 through 1622.

John Napier stated the needs for these inventions in 1614:

> "Seeing there is nothing (right well beloved Students of Mathematics) that is so troublesome to mathematical practice, nor doth more molest and hinder calculators, than the multiplications, divisions, square and cubic extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances."

Napier's logs to base 1/e 1594

Napier invented the basic strategy of going into easier addition to calculate difficult multiplication problems. He invented logarithmic tables to go into what he at first called "artificial numbers," add these together, and then take their sum into anti-log tables to come back to the multiplication product in natural numbers.

Three Napier misconceptions

There are three common misconceptions about Napier. First, Napier's logarithms were not multiplication chart distances, but were tables of numbers. Second, Napier's logarithmic tables were not to the base 10, but were to a base of 1/e, or 1/2.718, or .3673. Third, "Napier's rods" were not rods at all but were actually a cut and paste multiplication table.

Briggs' logs to base 10 1617

Henry Briggs, a professor at Gresham College, London, traveled to Edinburgh to convince Napier that his logarithms would be more useful if the Log 1 = 0 and Log 10 = 1. This is why the most used Logs to base 10 are often called "common logs" or "Briggian logs."

Napier accepted Briggs' suggestion.

Eshleman describes how to use logarithms to the base 10 in his article on pages 87-96 in this *Journal* issue (Eshleman, 1994).

Logs to base 2.718 are often called correctly "natural logs." However they are also often called incorrectly "Napierian logs." This is incorrect because Napier's logs were not to the base e (2.718), but to the base 1/e (.3679).

Logarithms only loosely related to Standard Celeration Chart

Logarithms are loosely related to the Standard Celeration Chart in only three ways. First, they both are methods for simplifying multiplication and division.

Second, Napier built his first table of logarithms to cover a range of natural numbers from 1 to 10,000,000 (16 base 2.718 cycles or 7 base 10 cycles). We built our Standard Celeration Chart to cover the full range of human performance frequencies from 1 to 1,000,000 per day (6 base 10 multiply cycles). This is only one x10 cycle shorter than Napier's selected range.

Third, the ignorance of common academic usage calls a chart like the Standard Celeration Chart with a multiply scale up the left and an add scale across the bottom "semi-logarithmic." This is a misnomer, for there is nothing logarithmic about the Standard Celeration Chart at all. It is not arithmetic, algebraic, nor digital. Rather, it is geometric and analogical.

What a log scale would look like

The top three cycles of the Standard Celeration Chart would look like this if they were logarithmic:

Logarithmic scale:	Multiply scale:
3.000 -----------	1000 ----------
.316 -----------	
2.000 -----------	100 ----------
.316 -----------	
1.000 -----------	10 ----------
.316 -----------	
0.000 -----------	1 ----------

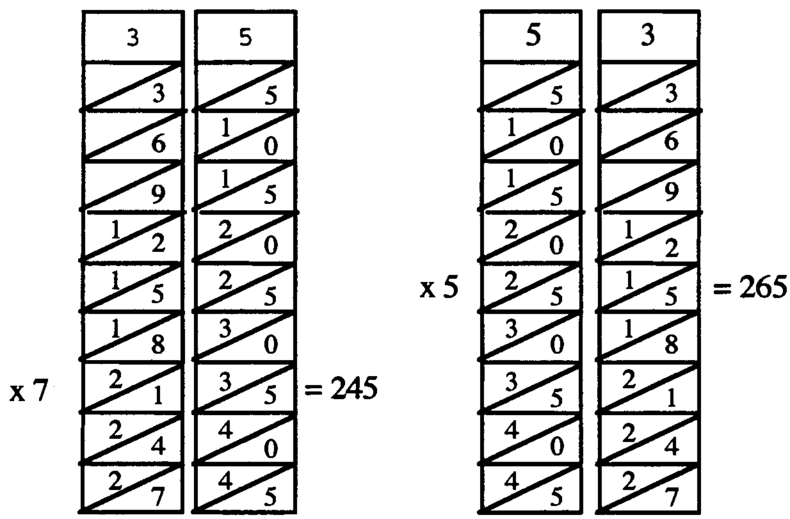
The counting lines would be evenly spaced, and the midpoint of each cycle would be a mantissa of .3164. Such a chart would be difficult to use.

Napier's Bones (cut and paste tables) 1617

It is clear that Napier was not fully satisfied with his cumbersome logarithms because he continued trying to simplify multiplication further. Napier invented his bones (sometimes called rods) in 1617, twenty three years after he invented logarithms.

Contrary to popular belief, Napier's bones were not linear, but were tabular. They were not lines with multiply distances like the slide rule, but were actually a nine column cut and paste table for rapidly calculating multiplication products. Each of the nine bones was a column of figures which could be easily placed along side any other bone. This easy abutting of any two columns permitted rapid addition across columns to get multiplication products of 1 to 9 times any number from 1 to 99.

Two examples of multiplying with Napier's bones follow. To multiply 7 times 35 we take the 3 bone and place it alongside and to the left of the 5 bone reading 35 across the top of both bones. Then we read down to the 7th row (times 7) and across the columns. In the upper left corner of the 7th row on the 3 bone, we find "2." We add the number "1" in the bottom right corner to the number "3" in the upper left corner of the 7th row on the 5 bone, getting the sum "4." Last we take the number "5" from the lower right corner of the 7th row on the 5 bone. These three numbers 2, 4, and 5 give us the product of 7 times 35 = 245! Very clever, right?



To multiply 5 times 53, we place the 5 bone to the left of the 3 bone, so we read 53 across the top of both bones. Then we read down to the 5th row (times 5) and across the columns. In the upper left corner of the 5th row on the 5 bone we find "2." We add the number "5" in the bottom right corner to the number "1" in the upper left corner of the 5th row on the 3 bone, getting the sum "6." Last we take the number "5" from the lower right corner of the 5th row on the 3 bone. These three numbers 2, 6, 5 give us the product of 5 times 53 = 265.

Napier's Bones faster than logs

Napier's bones were much faster to use than logarithms because there is no need to go into a table, convert to logarithms, add the logarithms, then go into an anti-log table to convert their sum back to the final product in natural numbers. The bones gave the multiplication product directly in one addition step with no conversions, but only with a limited range of whole numbers.

Napier's Bones not Standard Celeration C hart ancestor

Because Napier's columns were made of ivory or bone they were called "Napier's bones." Some called them "Napier's rods," but this mislead because they were rectangles, not rods. Calling them rods caused some careless historians to think they were linear distances like Gunter's line and were, therefore, precursors of the slide rule, which was not so. Because no distances were involved in Napier's bones, they were definitely not an ancestor of Gunter's line, Oughtred's slide rule, nor our Standard Celeration Chart.

Discovery of multiply nature of Standard Celeration Chart by teaching it simply

I had learned logarithms well as an undergraduate engineering student and had a part time job calculating metal surface areas and alloys for a major manufacturing company. I had even purchased an extra long 20 inch slide rule to get one more significant figure from the scales. I had used slide rules to calculate bomb and fuel loads as a flight engineer in the Army Air Corps. However, I did not understand multiplication and did not know why there was no zero, and why there were cycles on the slide rule.

It wasn't until we taught six-year-olds how to use the Standard Celeration Chart and described its features in plain English (Lindsley, 1991) that we began to understand the multiply scale.

It became clear that there was no zero on a multiply scale because zero isn't a multiply number. It isn't in the multiply world. When we must multiply and we encounter a zero (as in factorials), the rule is to put the closest multiply number to it in zero's place - the number one.

The cycles on a multiply scale are the counting cycles - a different add distance in each cycle. Going up the Standard Celeration Chart multiply scale, you count by ones in the first cycle, tens in the second, hundreds in the third, thousands in the fourth, ten thousands in the fifth, and hundred thousands in the sixth cycle. The nature of these counting cycles became crystal clear when we taught the Standard Celeration Chart by a Direct Instruction script (Maloney, 1982). [Editor's Note: See the revision of the Direct Instruction script by Cancio and Maloney in this issue, pages 15-45.

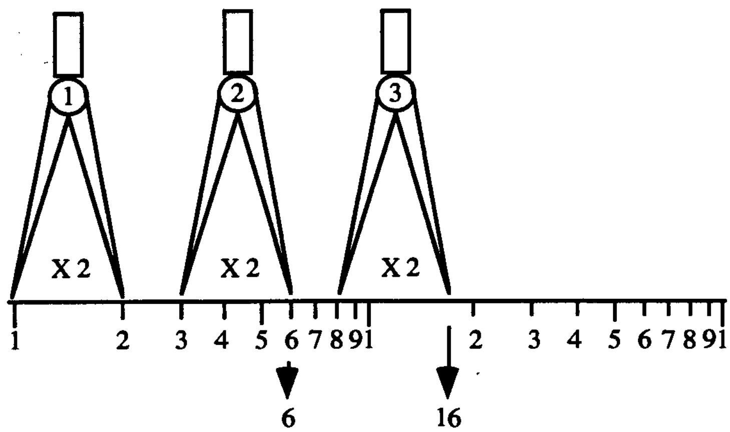
I found Napier's Bones and Gunter's Line this month

I had heard of Napier's rods and thought wrongly that they were multiply distances like Cusinier rods and precursors of the slide rule. It was not until asked to edit Eshleman's article (Eshleman, 1994), that I went into the libraries to do my homework and found Napier's Bones were really only cut and paste tables. Recently in the library, I discovered Gunter's line, the true precursor of the slide rule and our Standard Celeration Chart.

Gunter's multiplying Line 1620

Edmund Gunter invented a line with equal multiple *distances* in 1620. Multiplication products were found easily by taking a *distance* with a mechanical drawing divider and physically adding it to any number on the Line. Equal distances on Gunter's line are equal multiples.

For example, the divider's left point is put on 1 and the right point adjusted to the multiplier *distance* "x2." See divider 1 below. This *distance* is now x2, and can be applied to any other number on the line.



For example, to multiply 2 times 3, we place the left point of the x2 divider on 3, and read the product 6 opposite the right divider point. See divider 2 above. To multiply 2 times 8, we place the left point of the x2 divider on 8, and estimate the product 16 on the line opposite its right divider point. See divider 3 above. To divide you subtract the distances with the divider.

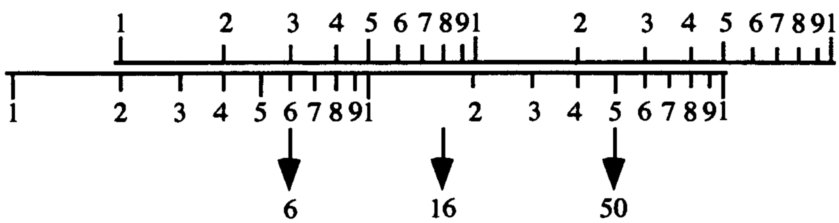
The Standard Celeration Chart multiply scale is exactly like Gunter's multiply line. When we make two tick marks on a card to take a multiplier distance from the Standard Celeration Chart, we do what Gunter did when he adjusted his divider length. When we place the bottom tick on our card across from the number we want to multiply on the Chart, we do what Gunter did when he placed his left divider point on the line number that he wanted to multiply. When we read our product across from the top tick mark on our card, we do what Gunter did when reading his product where the right divider point touched his multiply line.

For this reason I always call the scale on the Standard Celeration Chart a "multiply scale," rather than a "logarithmic scale." It is truly a scale of multiply distance like Gunter's line. It is not a logarithmic scale, even though commonly called so in error. Our Standard Celertion Chart has no characteristic, no mantissa, and the divisions are not separated by equal add distance as would logarithmic mantissae.

Two experts on graphs have written that charts with a multiply scale up the left and an add scale for time across the bottom should not be called "semi-logarithmic." Rather they suggest calling them "rate of change" charts (Karsten, 1923; Schmid, 1951). "Rate of change" in frequency (number per day) is celeration (number per day per week).

Oughtred's slide rule 1622

Two years later William Oughtred added a second multiply line to Gunter's multiply line which slid along the top of the first line from which the multiplicand and the product could be read. This second line accomplished what Gunter did when adjusting his dividers and moving them. The second sliding line was easier to use and more reliable than Gunter's dividers. The sliding of the second line gave it the name " slide rule."



To multiply using Oughtred's slide rule, we set the left index of the top line to the multiplier "times 2". Now the product of 2 times every number in the top line is directly below it in the bottom line. Directly below 3, we find 6. Directly below 8, we interpolate 16. Directly below interpolated 25, we find 50.

Proof that this is as important a discovery as Napier's logarithms is attested to by the fact that there are 13 references under slide rules and only 11 under logarithms in the University of Kansas Library System.

Using a card to multiply or divide on the Standard Celeration Chart is like using the divider on Gunter's multiply line. Using a frequency finder (often called the "slider" by school children) to divide the count by the number of minutes on a Standard Celeration Chart is like using a second multiply scale as did Oughtred when he built his first slide rule.

In 1675 Isaac Newton used three parallel scales to solve the cubic equation and invented the sliding indicator seen on all modern slide rules (Kells, Kern & Bland, 1939).

Yugoslavian and Russian graphical statistics

More Eastern European texts describe using graphical statistics than English and American texts. Novak (1967) in his Yugoslavian text described in great detail how to use dividers and Gunter's Line to solve the full range of algebraic problems. Gerchuk (1968) in his Russian text demonstrated the frequency slider or "finder" that we independently developed as a Standard Celeration charting tool. Gerchuk also demonstrated how to use a percentage slider to read percentage distances from a multiply scale chart (Gerchuk, 1968, p.131).

Graphical statistics ignored in US until 1970s

Using graphics for serious data analysis was ignored in the United States from 1930 to 1970 (Tufte, 1983, p. 53). There was, and still is, a strong bias in favor of numerical formula-based parametric statistics. High speed computers have made calculating formulae so easy that most who use them do not know the formulas, nor understand the results. In the late 1960s, John Tukey made statistical graphics respectable (Tukey & Wilk, 1970; Tukey, 1977).

Historians separate logs from slide rules

Stephen Graf (1994) called my attention to Graham Flegg's excellent text, *Numbers, their history and meaning*. (Flegg, 1983). Here Flegg points out the difference between calculating and computing. Napier with his logarithms are in chapter four on calculating. Napier's bones and Gunter's multiply line along with Oughtred's slide rule are in chapter five on computing. Here we find a top historian of mathematics separating Napier's logarithms from Gunter's line.

Tables of logarithms show add distance

One reason that we have never seen multiplication as a distance before Gunter's line is that tables of logarithms show add distance. In a table of logarithms there are 100 entries (ten rows of ten mantissae each) between 10 and 20. That is 100 entries for a x2 multiplier. Between 50 and 100 there are 500 entries - five times the distance of 10 to 20! But, because 50 to 100 is a x2, there should be the same number of entries if the log table showed multiply distance. But it doesn't! It shows add distance. There are exactly the same number of entries between 20 and 30 as between 10 and 20. The number of entries defines the space taken and the distance in the table. It is add distance.

Therefore, the logarithmic table contains unnatural numbers that can be used to get the products of multiplication, but the *distances* are equal add. There are the same number of entries between 10 and 20 (100) as between 80 and 90 (100). Because the number of entries marks the distance, the distance is add, even though the entries carry multiply values.

This is why no one saw multiplication as a *distance* until Gunter's line, Oughtred's slide rule, and our Standard Celeration Chart. Also few could see the relationships and differences between multiplication and addition until they could see them both as differently marked off distances on lines.

Multiply scale and standard view

Our Standard Celeration Chart is based on a geometric multiply scale which accomplishes geometrically what logarithms accomplish arithmetically. Our Chart is also based on a standard view which is not old, but very new. There are only a few standard view charts. The standard audiogram (Montgomery, 1932) for displaying hearing loss is one of the rare examples. Another rare example is Skinner's standard "cumulative response record" (Skinner, 1938; 1976). Skinner described it by its left hand vertical scale. If described by what its slope displayed, it would have been called a "standard frequency chart" (Lindsley, 1994). Standard charts are so rare that there is not yet a standard videogram for displaying visual loss.

Celebrate 375 years of Gunter's Line

1995 will be the 375th anniversary of the invention of Gunter's line in 1620. Gunter's line showed multiplication and division as physical distances to the world for the first time. Prior to Gunter's line, the world had only seen addition and subtraction as distances on rulers and yard sticks. All Standard Celeration Chart users should honor Gunter and celebrate the 375th anniversary of his line in 1995.

References

Auman, J. (1994). Personal letter, 3 October.

Eshleman, J. W. (1983). All the known Precision Teaching references. *Journal of Precision Teaching, 4* (3), 56-67.

Eshleman, J. W. (1990a). The history and future of Precision Teaching. *Journal of Precision Teaching, 7,* (2), 18-27.

Eshleman, J. W. (1990b). Precision Teaching reference list. MWD-PTRF 90.3 Macintosh Word Document Disk, Precision Media, Kansas City KS.

Eshleman, J. W. (1994). Celebrating 400 years of logarithms. *Journal of Precision Teaching, 12,* [in press].

Flegg, G. (1983). *Numbers, their history and meaning.* New York: Barnes and Noble.

Gerchuk, I. P. (1968). *Graficheskie metodi b statistike.* Moskba: Statistika.

Graf, S. A. (1994). Telephone conversation, 11 September.

Karsten, K. G. (1923). *Charts and graphs.* New York: Prentice Hall.

Kells, L. M., Kern, W. F., & Bland, J. R. (1939). *The log log duplex trig slide rule.* New York: Keufel & Esser Co.

Lindsley, O. R. (1991). From technical jargon to plain English for application. *Journal of Applied Behavior Analysis, 24,* 449-458.

Lindsley, O. R. (1992). *Skinner on measurement:* Kansas City, KS: Precision Media.

Maloney, M. (1982). Teaching the daily Standard Behavior Chart: A Direct Instruction approximation. *Journal of Precision Teaching, 2,* (4), 11-30.

Montgomery, H. C. (1932). Do our ears grow old? *Bell Laboratory Records, 10,* 311.

Napier, J. (1614). *Wonderful canon of logarithms.*

Napier, J. (1617). *Rabdologia.*

Novak, A. (1967). *Metodi i tehnika graficke analize odnosa.* Belgrad: Primena.

Schmid, C. F. (1979). *Handbook of graphic presentation.* New York: Ronald Press.

Skinner, B. F. (1938). *The behavior of organisms.* Englewood Cliffs, NJ: Prentice-Hall.

Skinner, B. F. (1976). Farewell, my LOVELY! *Journal of the Experimental Analysis of Behavior, 25,* 218.

Tufte, E. R. (1983). *The visual display of quantitative information.* Cheshire, CT.: Graphics Press.

Tukey, J. W., & Wilk, M. B. (1970). Data analysis and statistics: techniques and approaches. in E. R. Tufte (Ed.), *The quantitative analysis of social problems.* Reading, Mass.: Addison-Wesley.

Tukey, J. W. (1977). *Exploratory data analysis.* Reading, Mass.: Addison-Wesley.

How about JPT on-line on INTERNET? Plans are underway to summarize articles of each issue. Any ideas or contributions on how best to accomplish such a world-wide exposure of PT???

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