Celebrating 400 Years of Logarithms

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Upon encountering Precision Teaching for the first time, a person will soon come into contact with an unusual type of graph paper. The graph has light blue grid lines arranged in "landscape" view on white paper. What makes the graph unusual for most people though, is the arrangement of the horizontal lines going across the paper. More specifically, the scale up the left seems strange. Instead of the familiar equally spaced intervals marked off, the lines show an odd pattern. Starting at the bottom of the Chart at .001 and going up, the lines get closer and closer together. When one reaches .01, and continues up, the pattern repeats. The same pattern starts over at .1 and then again at 1, 10, and 100. How odd! Most people quickly get past this strangeness and soon learn to plot dots, read data, and learn the lingo that goes with the graph.

One quickly learns that people in Precision Teaching do not use the word "graph." The blue graph is called a "Chart." In fact, one hears it called "the Chart." Further, the Chart has a name: Standard Celeration Chart. One also learns to call that strange scale up the left side a "multiply-divide" scale. Sometimes one may hear the Chart referred to as the "six cycle chart," and even on occasion as a "semi-log chart." One may even learn that semi-log means that one of the two axes is a logarithmic or log scale. One does not need to know what "log scale" means in order to work with the Chart. Most people probably just file that bit of information away anyway.

But what if one wants to know what "logarithmic" means? What is a log scale? Simply put, it is a scale constructed from a table of logarithms (e.g., Schmid, 1954). You may have heard of logarithms along the way, but never really learned what they are, how to work with them, or what their relationship to the Chart happens to be. Knowing about logarithms may help explain that strange log scale and provide insight about why, for instance, there are "six cycles," or why the distance from 1 to 2 equals the distance for 3 to 6, 5 to 10, 40 to 80, and so on. Besides, it is the quadrennial of logarithms this year. So why not find out about the logarithms that underlie the Chart's log scale?

Background

Logarithms were invented by John Napier (1550 - 1617), of Edinburgh, Scotland. Napier first got the idea for logarithms in 1594, four hundred years ago (Hellemans & Bunch, 1988; Hobson 1914). However, Napier did not publish his invention for another 20 years. In 1614, he wrote *Mirifici Logarithmorum Canonis Descripto*. The Latin title translates to "Description of the Wonderful Canon of Logarithms." The book explained logarithms, and gave us the first log tables and rules for using logarithms.

Holding title as the 8th Laird of Merchiston, Napier inherited his father's estate, and spent his life as a land-owning proprietor, indulging in inventing, theology, and mathematics. Thirty-eight at the time of the ill-fated Spanish Armada (1588), Napier spent several years inventing a set of then-fantastic military devices for repelling the Spaniards. Though mainly never built, his inventions included an array of burning mirrors similar to those devised by Archimedes in ancient times, artillery designed to destroy all life within the radius of one mine, and round armored chariots (Hobson, 1914). He also occupied his time publishing anti-Catholic tracts.

By 1594, Napier's interests turned to a new way of calculating, using numbers expressed in exponential form. First called "proportionate numbers," but later "logarithms," Napier spent the next 20 years pursuing this mathematical invention, eventually devising tables of logarithms. Napier used a number close to 1/e (where e =2.7182818284... a repeating decimal) as the base for his logarithms (Hobson, 1914). Logarithms to base e are called "natural" logarithms, or "Napierian" logarithms. Often, Napier is miscredited for inventing natural logarithms. Instead, his logarithms pertained to trigonometric functions, specifically the sines of angles for 0^o to 90^o. These logarithms proved especially applicable to astronomy -- and hence to naval navigation. The mercantilist British East India Company, formed in 1600, needed such inventions to provide an emerging imperialist global power with a competitive edge.

Publication of Napier's book in 1614 provoked an immediate stir among scientists and mathematicians of the day, among them the famous astronomer, Johann Kepler. Another was Henry Briggs, a Fellow at St. John's College and Gresham Professor of Geometry in the City of London. In 1615, Briggs visited Napier in Edinburgh for one month, and then again in 1616. Briggs proposed that logarithms be developed using the more familiar and convenient base 10. Napier agreed. They called Brigg's idea "improved logarithms" (Hobson, 1914). Briggs calculated these improved logarithms, and in his 1616 visit showed Napier the tables he had made. Then in 1617, the year Napier died, in *Logarithmorum chilias prima* Briggs published a 16 page table of what came to be called "common" logarithms. Later, in 1624 Briggs published *Arithmetical logarithmica*, which had common logarithms to 14 places, covering numbers from 1 to 20,000 and from 90,000 to 100,000 (*Encyclopedia Americana*, 1993, Volume 4, p. 554).

Though there are several kinds of logarithms, the ones that will concern us here will be the Briggsian common logarithms. Again, common logarithms are set up in base 10. A table of common logarithms has been included in Appendix I, which will be referenced several times.

Defining and Finding Logarithms

So, what are logarithms? First, every number has a logarithm associated with it. Every number can be converted to its logarithm, and every logarithm can be converted back to a number. Second, logarithms are set up on a multiply-divide basis. Instead of multiplying two numbers and obtaining a product, one can convert the numbers to their respective logarithms, add the two logarithms, and convert this sum back to a number. The number would be the same as the product. Logarithms became more useful when raising a number to a power or finding the nth root of a number. To raise a number to a power, one multiplies the exponent -- the power -- by the log of the number. Thus, 8^{32} would be 32 times the log of 8, for instance.

As mentioned, every number can be converted into a logarithm. A logarithm is another number, but with a specific definition. The definition focuses on the decimal point of the logarithm. For example, the logarithm for the number 80 is 1.9031. Every logarithm has two parts. These parts are the "characteristic" and the "mantissa." A simple rule keeps this terminology clear (e.g., see Bruhns, 1939). In a logarithm, the characteristic is that part of the logarithm to the left of the decimal point. The mantissa is that part to the right of the decimal point. Thus, for the logarithm 1.9031, the characteristic is 1, and the mantissa is .9031.

There are precise rules for finding the characteristic and mantissa for any number. For any whole number, the characteristic is always one less than the number of digits to the left of the decimal point of the whole number. The number 80 in the example is really 80.0 and thus has two digits to the left of the decimal. The characteristic is one less than the number of digits to the left. Thus, the characteristic for 80 is 1. The characteristic for 800., which has three digits to the left of the decimal, would be 2 according to the rule. The characteristic for 8 would be 0. because 8 is only one digit. For numbers equal to 1 or greater, the characteristic is always positive.

For decimal numbers, the characteristic is preceded by a minus sign. Another rule exists for finding the characteristic of decimal numbers. For decimal numbers -- those beginning with a decimal point -- the characteristic is the number of places to the right of the decimal to the first significant (non-zero) digit. For the decimal number 0.8, the characteristic would be -1. The first place to the right of 0.8 has a significant digit. For 0.08, the characteristic would be 12, and for 0.008 it would be -3. Table 1 illustrates characteristics and the respective ranges of numbers they cover.

Table I

Characteristics and the Number Ranges They Cover

Characteristic:	Number Range Covered:		
5	100,000	-	999,999+
4	10,000	-	99,999+
3	1,000	-	9,999+
2	100	-	999+
1	10	-	99+
0	1	-	9+
-1	.1	-	.9+
-2	.01	-	.09+
-3	.001	-	.009+
-4	.0001	-	.0009+
-5	.00001	-	.00009+

A person experienced with the Standard Celeration Chart will see at a glance that the characteristics relate directly to the six cycles of the Chart. Each cycle on the Chart corresponds to a range of numbers. Each cycle thus corresponds to a particular characteristic. One can use Table 1 to figure out the characteristic for any frequency on the Chart. For example, 8 per minute falls within the 1 - 9 range. The numbers in this range have one digit, so we know that their characteristics are 0. However, 12 per minute would have a characteristic of 1; 35 per minute would also have a characteristic of 1, as would 67 per minute and even 99 per minute. 100 per minute would have a characteristic of 2. Going in the other direction on the Chart, 0.2 per minute would have a characteristic of -1, whereas 0.05 per minute would have a characteristic of -2.

Characteristics are that portion of a logarithm that explain the six cycles. The mantissas, on the other hand, explain why within each cycle the lines get closer and closer together as you move up the scale. Mantissas are the other half of logarithms. They can be found in Appendix 1.

In logarithms, the mantissa follows the decimal point. Mantissas, however, are independent of the decimal points of the number being converted to a logarithm. The characteristic tells one which cycle he is in. The mantissa tells one where one is within any cycle. The cycles all have the same mantissas and differ only according to the characteristics. This may sound like a confusing distinction, and some examples may help clarify. Thus, for example, 800, 80.0, 8.00, 0.800, 0.0800, and 0.00800 all have the same mantissa, but different characteristics. The mantissa for 800, 8.00, and 0.00800 is the same: The mantissa for both 40.0, and 4.00 is .6021. 40.0 has two digits to the left of its decimal, so its characteristic is 1. 4.00 has only one digit to the left of its decimal, so its characteristic is 0. The log of 40 is 1.6021, and the log of 4 is 0.6021.

Using the table in Appendix 1, one can find the logarithms for 10, 20, 30, 40, 50, 60, 70, 80, and 90. For all of these numbers, the characteristic is 1. So the table in Appendix 1 is used to find the mantissas. Starting with 10, one should scan down the column that has N as the header. The first number in this column is 10. Next, look to the first column to the right. Its header is 0. This means one is finding the mantissa for 10.0 under this column, which is 0000. Continuing on the same line to the right, the mantissa for 10.1, then, is .0043. The mantissa for 10.2 is .0086. One could keep reading across this line until one came to the column headed with 9. The mantissa for 10.0 is .0374.

If one moves down column N, the next number down is 11. The mantissa for 11.0 is .0414. The mantissa for 12.0 is .0792. On down column N, one finds the mantissa for 20.0 is .3010. For 30.0 the mantissa is .4771, and for 40.0 it is .6021. 50.0 has a mantissa of .6990; 60.0 a mantissa of .7782; 70.0 is .8451; 80.0 is .9031; and 90.0 is .9542. Bear in mind, again, that mantissas are independent of the decimal location of the original numbers. The mantissa of 9.00 is also .9542, for example. The full logarithm for 9.00 is 0.9542, and the full logarithm for 90.0 is 1.9542.

One may notice something else about the pattern of these mantissas. The mantissa for 9.00, which is .9542, is not very much above the mantissa for 8.00, which is .9031. The mantissa for 2.00, which is .3010 on the other hand, occurs a considerably larger difference up from the mantissa for 1.00, which is .0000. Note that the increments in mantissas get smaller and smaller as the numbers increase. Indeed, in examining Appendix 1, one should note that the mantissas of all the numbers from 80 through 99 begin with 9. It is interesting to compare that to how many begin with 0 or 1.

Another way of thinking about mantissas would be to consider them proportions, of the way up, the number scale. The lowest mantissa, 0000, is 0% of the way up. .3010 is nearly a third of the way up. .6990, the mantissa for 50.0, is more than half the way up; indeed, this is more than two-thirds the way up. Look at a Standard Celeration Chart. Five per minute is certainly more than two-thirds the way up between 1 per minute and 10 per minute. Eight per minute is 90% of the way up that cycle, and 9 per minute is 95% up that cycle.

Table 2 lists the common logarithms for the top three cycles of the Standard Celeration Chart.

Table II.

Logarithms of Frequencies of the Top Three Cycles of the Standard Celeration Chart

Count per Minute:	Logarithm:
1000	3.0000
900	2.9542
800	2.9031
700	2.8451
600	2.7782
500	2.6990
400	2.6021
300	2.4771
200	2.3010
100	2.0000
90	1.9542
80	1.9031
70	1.8451
60	1.7782
50	1.6990
40	1.6021
30	1.4771
20	1.3010
10	1.0000
9	0.9542
8	0.9031
7	0.8451
6	0.7782
5	0.6990
4	0.6021
3	0.4771
2	0.3010
1	0.0000

Equal Differences Represent Equal Distances

Examine Table 2 again. Notice the log of 2 is .3010. The log of 4 is .6021. The difference is 3011. Now notice the log of 8, which is .9031. The difference between the log of 8 and the log of 4 is .3010 -- which happens to be equal to the log of 2. Also, .3010 is practically the same as .3011. On the Chart, the distance from 2 to 4 is the same as the distance from 4 to 8, the same as the logs. Consider a couple more examples. The log of 3 is .4771, and the log of 6 is .7782. The difference is .3011, the same as the log of 2. To go from 3 to 6, one multiplies by 2. On the Chart, the distance from 3 per minute to 6 per minute is the same as that between 1 per minute and 2 per minute. The same results if one considers 5 per minute and 10 per minute. The log of 5 is 0.6990, and the log of 10 is 1.0000. The difference is 0.3010 -- again equal to the log of 2! On the Standard Celeration Chart, equal distances represent equal "multiplies by" or equal "divides by" (Pennypacker, Koenig, and Lindsley, 1972).

Working With Logarithms

One more term should be introduced: An antilogarithm, or antilog for short, is the number that corresponds to a given logarithm. The process of converting a logarithm back to a number is referred to as finding the antilog. To find the antilog of 2.9031, one first consults Appendix 1 and finds the mantissa .9031. It is in the 0 column next to 80. The characteristic is 2, so one knows that there will be three digits to the left of the decimal. The antilog of 2.9031, then, is 800. Technically, the numbers listed up the left scale of the Standard Celeration Chart would be antilogs (Schmid, 1954); however, it is simpler to treat them as natural numbers on a multiply scale.

One may notice that the table in Appendix 1 does not actually list all possible logarithms. For example, it lists the mantissa for 10.0, which is 0000, and the mantissa for 10.1, which is 0043. But what about 10.05, which is between 10.00 and 10.10? For this number, or any other "in between" number anywhere else, a process of interpolation must be used to compute the logarithm. Interpolation must also be used if the mantissa of the log is to be converted back to a number that is not in the table. The log 0.5484 is not in the table. But one can find .5478 and .5490 in the table and use interpolation to find the antilog. Suffice to say, the antilog of 0.5484 is the number 3.535.

One can multiply and divide numbers using their logarithms. To multiply 6 x 7, one would add their logs, 0.7782 and 0.8454 and get 1.6233. One will not find .6233 in Appendix 1, but will find .6232, next to the number 42. There are some slight inaccuracies when working with logarithms, for it is known that 6 x 7 equals 42, and not an approximation to 42.

A number can be raised to a given power by using logarithms. To find 20^3, for example, one would multiply the exponent, 3, by the log of 20. The log of 20 is 1.3010. 3 x 1.3010 equals 3.903. The mantissa .903 is not in Appendix 1, but is close to .9031, which is the mantissa for 80.0. One would convert 3.903 back into 8000. Again, there are some slight inaccuracies.

Finally, one can find any nth root of a number using logarithms. To find a root, divide the logarithm of a number by the index of the root and find the antilog. Thus, to find the 4th root of 60.0, one would first find the log of 60.0, which is 1.7782. 1.7782÷4 equals 0.44455. The antilog of 0.44455 would be 2.789 approximately. Again, small inaccuracies accrue.

Conclusion

Logarithms are 400 years old and serve as the basis for the vertical scale on the Standard Celeration Chart. Logarithms are not needed to work with the Chart. They are seldom taught anymore. Slide rules, which were based on logarithms, are also a thing of the past. Today, neither logarithms nor slide rules are needed to find products, quotients, powers, or nth roots. Hand-held calculators can do the job faster and with more precision. But logarithms came before calculators, and if we want to know our background, we may want to know something about logarithms.

References

Bruhns, Dr. (1939). *A new manual of logarithms to seven places of decimals.* (18th ed.) Princeton, NJ: D. Van Nostrand. (Original work published 1869)

Encyclopedia Americana. (1993). Intentional Edition. Danbury, CT: Grolier.

Hellemans, A., & Bunch, B. (1988). *The timetables of science. A chronology of the most important people and events in the history of science.* New York: Touchstone.

Hobson, E. W. (1914). *John Napier and the invention of logarithms, 1614. A lecture.* London: Cambridge University.

Pennypacker, H. S., Koenig, C., & Lindsley, O. R. (1972). *Handbook of the standard behavior chart.* Kansas City, KS: Precision Media.

Schmid, C. F. (1954). *Handbook of graphic presentation.* New York: The Ronald Press.

Plan to attend an open meeting of the *JPT* Editorial Board at ABA '95! Bring ideas and nominations for the Board!

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Appendix 1

Table of Common Logarithms (Mantissas)

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522

45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289

85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9898	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996