Fisher's Exact Probability and Precision Teaching: Uses, Limits and an Efficient Method of Calculation.

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Workers in the discipline of Precision Teaching occasionally need to make statistical statements about their data. When these occasions arise, providing an exact probability is a more precise statement than providing an estimate of a significance level (Lindsley, 1980). The purposes of this paper are to refresh the Precision Teacher's memory of Fisher's exact probability formula; to show how to efficiently calculate exact probability; to provide an example of the use of this statistic with data plotted on the Standard Behavior Chart; and to share some of the strengths and weaknesses of Fisher's exact probability test which have been expressed in the statistical literature.

It should be made clear that the author is not advocating statistical analysis, but simply recognizes that the Precision Teacher must communicate with related professionals. Some of these professionals need to see or hear a significance level stated before they appear able to process the data one wishes to share. If the methodology and product of Precision Teaching is not compromised by the dictates of statistical analysis, the only possible loss is the time used to punch numbers into a calculator and record the product.

The frequencies (count per minute, etc.) generated over time (successive calendar days, etc.) and recorded on the Standard Behavior Chart can be cast into a 2 x 2 contingency table. The count per minute, the y axis, can be bisected by the median frequency of the distribution for the frequencies charted. Successive calendar days, the x axis, can be divided at a phase change. There are a number of other rational methods for dividing the x and y axis of the Standard Behavior Chart into the four quadrants necessary for the construction of a 2 x 2 contingency table.

Chart 1 provides an example, using the median frequency to bisect the y axis and the phase change to divide the x axis. Each dot on the chart represents the middle frequency counted per day of appropriate linguistic pauses which Steve emitted while vocally describing a variety of stimuli. Three one-minute timings were obtained per session. The frequencies to the left of the phase change line (Phase I) were obtained when the clinician provided pictures and children's books for Steve to describe. After the phase change (Phase II) the stimuli which the clinician provided were manipulable objects. The aim of the training was to increase Steve's count of appropriate linguistic pauses to at least 12 per minute (Flanagan, 1977). The Phase II change was made because the Phase I was not yielding the desired celeration.

Employing the data from Chart 1, the median frequency and the phase change were used to assign each frequency plotted to one of the four cells of a 2 x 2 contingency table. The median frequency for the two phases combined was equal to 10. The phase change occurred before the 50th successive calendar day. For Phase I the frequencies were assigned to cell a or c as follows: If a frequency was above the median it was assigned to cell a. If the frequency was at or below the median it was assigned to cell c. The same

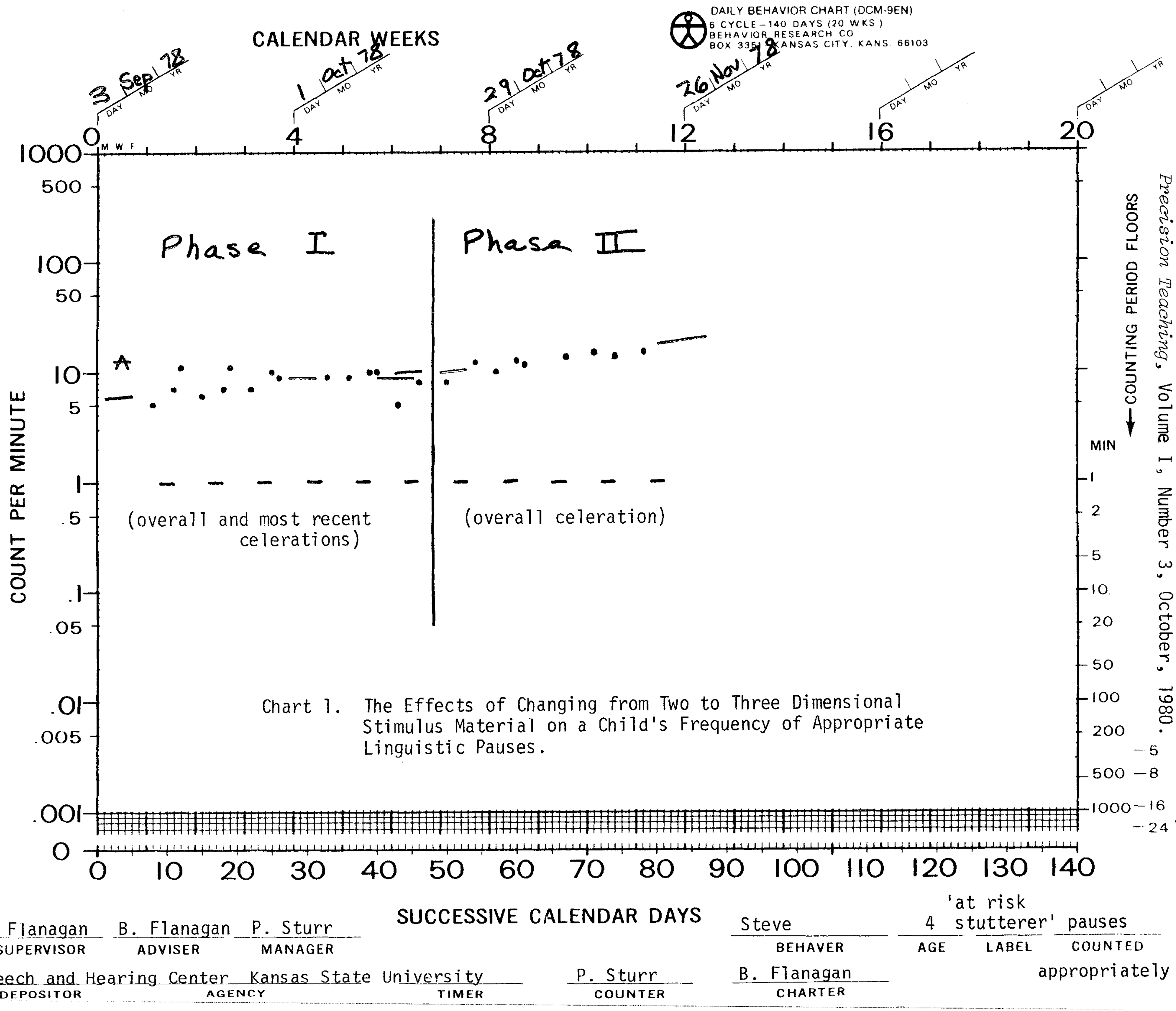

Chart 1. The Effects of Changing from Two to Three Dimensional Stimulus Material on a Child's Frequency of Appropriate Linguistic Pauses.

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criteria were used for assigning the frequencies in Phase II to cells b or d. The data from Chart 1 casted as described into 2 x 2 table are shown in Table 1.

Table 1

Frequencies from Chart 1 Divided by Median and Phase Change
Cast Into a 2 x 2 Contingency Table

	Phase I	Phase II	
Above MD MD = 10	a 2	b 7	a + b = 9
At or Below MD	c 13	d 2	c + d = 15
	a + c = 15	b + d = 9	N = 24

A commonly cited working formula (Cochran, 1952; McNemar, 1955; Siegel, 1956) for computing Fisher's exact probability test (1934) is

$$P = \frac{(a+b)!(c+d)!(a+c)!(b+d)!}{N!\ a!\ b!\ c!\ d!}$$

or

using the data from Chart 1

$$P = \frac{(9!) \times (15!) \times (15!) \times (9!)}{(24!) \times (2!) \times (7!) \times (15!)\ (2!)}$$

Statisticians agree the test cited is laborious to compute. Recall that a factorial (!) is the product of a given series of consecutive whole numbers beginning with 1: as the factorial of 5 is 5 x 4 x 3 x 2 x 1 or 120. Consider the following relation between n, n! and the common logarithm of n! (log n!) shown in Table 2.

Table 2

Factorials and Their Common Logarithms

n	n!	log n!
0	0 (by definition)	0.00000
1	1	0.00000
2	2	0.30103
3	6	0.77815
4	24	1.38021
5	120	2.07918
6	720	2.85733
7	5040	3.70243
8	40320	4.60552
9	362880	5.55976
10	3628800	6.55976

Given the size of the numbers which need to be computed in the problem at hand, one gains an intuitive understanding of why Fisher's exact probability test is considered laborious to compute. Even with the calculators available today, the capacity of these machines is quickly exhausted. However, Finney (1948) suggested the labor can be reduced by the use of a table of the common logarithms of the factorial functions. It can be noted in the right column of Table 2 how the conversion of factorials to a common logarithm keeps the numbers to a manageable size. Even the log of 100! is 157.97000, a number which is not beyond the capabilities of today's electronic calculators.

To turn the laborious task of computing Fisher's exact probability into a relatively modest effort, the Precision Teacher needs to be reminded of two rules which are required for the multiplication and division of logarithms:

1. To multiply two numbers, add their logarithms.
2. To divide two numbers, subtract their logarithms.

These mathematical relations hold regardless of the magnitude of the number.

The conversion of the linear equation cited previously for computing Fisher's exact probability to a logarithmic equation requires that the two rules stated above be followed. The resulting logarithmic equation is:

$$\log P =$$

$$[\log(a+b!)+\log(c+d!)+\log(a+c!)+\log(b+d!)]-[\log(N!)+\log(a!)+\log(b!)+\log(c!)+\log(d!)]$$

$$P = (\log P)^{10}$$

or

using the data from Chart 1

$$\log P = \begin{array}{l} \log(9!) = 5.55976+ \\ \log(15!) = 12.11650+ \\ \log(15!) = 12.11650+ \\ \log(9!) = 5.55976 \end{array} \quad - \quad \begin{array}{l} \log(24!) = 23.79271+ \\ \log(2!) = 0.30103+ \\ \log(7!) = 3.70234+ \\ \log(2!) = 0.30103 \end{array}$$

The sums of the above logarithmic conversions are:

$$\log P = 35.35253 - 37.89148 = -2.53895$$

$$P = (-2.53895)^{10} = .00289$$

If the reader has a calculator equipped with the factorial (x!) common logarithm (log) and antilogarithm (x^{10}) math function keys, plus a memory capability, the computation of the above data takes a few minutes. See the Appendix for the exact steps necessary, with a calculator equipped as described, for the calculation of the product of this equation.

To obtain a probability (P) value comparable to the usual significance test of the null hypothesis, all P's more extreme than the obtained P must be calculated and added to the P calculated above. This can be made clear by an example. In Table 3 the observed set of frequencies (part I) and sets showing more extreme frequencies are found (parts II and III).

Table 3

The Series of 2 x 2 Table Required
For Calculation of P Directly and Exactly

I
observed

a 2	b 7	a+b =9
c 13	d 2	c+d =15
a+c =15	b+d =9	N=24

II
more extreme

a 1	b 8	a+b =9
c 14	d 1	c+d =15
a+c =15	b+d =9	N=24

III
more extreme

a 0	b 9	a+b =9
c 15	d 0	c+d =15
a+c =15	b+d =9	N=24

Note that each part is based on the preceding part by subtracting 1 from a and d and adding 1 to b and c. This process is continued until cell a or d or both are equal to zero. The ad bc subtraction and addition process would be reversed if a zero cell could be reached in fewer steps by progressing in the reverse direction.

Part I contains the observed frequencies presented in Table 1 and calculated in the preceding section using a logarithmic formula for calculation of P. Thus P = .00289. Table 3 shows that the marginal frequencies for Parts II and III remain unchanged from Part I. From the viewpoint of efficiency of calculation, this fact is important because the value obtained for the left half of the calculation [log(a+b!)+...] - remains a constant in the calculations of the P's for Parts II and III. The resulting probabilities are:

$P^{I} = .00289$
$P^{II} = .0001033$
$P^{III} = .0000008$
$P = P^{I} + P^{II} + P^{III} = .002995 = .003$

Without any assumptions or approximations, which would be necessary with alternative statistical treatments, the contingency table observed may be judged to significantly contradict the null hypothesis of proportionality if P is a small quantity: in this case, .003, or about 1 out of 333, showing that for any case in which the hypothesis were true, observations of frequencies obtained would be extremely exceptional (Fisher, 1934). Using an alternative statistic, Chi-square, the best the Precision Teacher could do would be to estimate the P value. Instead of stating that the probability of proportionality of the Phase I versus Phase II frequencies could be explained by chance one time out of 333 times, it would be necessary to state that the probability of proportionality was greater than one time out of 100, but less than one time out of 1000.

The Precision Teacher should recall that Fisher's exact probability test is a one-tailed test. That is, one must be willing to state the expected direction of change in frequency that the conditions of the phase change are likely to produce. The expected direction of change in frequencies between Phase I and II in this example was that the frequency would increase. If the Precision Teacher does not have a basis for expecting what effect the phase change will have on subsequent frequencies, such as with a maintenance schedule, a two-tailed test of probability should be used. To obtain a two-tailed probability value, multiply the P obtained above by two.

Fisher recommends the test for all types of dichotomous data; this recommendation has been questioned. Tocher (1950) has offered a modification to correct for possible Type I errors which would be useful in certain sampling situations. According to Cochran (1952), Tocher's modificiation should be used when:

(1) One is selecting a random sample (n) from some population and classifying the observation into one of four cells.

(2) One is taking a random sample from a population denoted by A, and an independent random sample size r^2 from another population.

In the example in this report, neither of these situations appear to fit. The frequencies from Phase I and Phase II have not been randomly selected, but represent the population of frequencies which exist for Steve at the defined task. The population of interest is Steve's rate of appropriate linguistic pauses and the related variable of his fluency while speaking. Unlike Fisher's example (1956) of "Mathematics of a lady tasting tea," the question of whether the frequencies of Steve's training data will relate his behavior in another setting is not a theoretical issue, but an empirical question, which can be and was resolved by the collection of repeated probe data (Sidman, 1960) in settings removed from the training situation.

The probability statement concerning the frequencies in Phase I versus Phase II will most likely have the greatest utility when the author is attempting to share data with a related professional who has a strong history of reinforcement for seeing or hearing a significance level stated before he can look at data.

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APPENDIX

The author went to the University Book Store and searched for the most inexpensive name brand calculator which contained the factorial (x!) common logarithm (log) and antilogarithm (x^{10}, usually) math functions and a memory to accumulate the log conversions. A calculator which met these requirements was identified (Texas Instruments, Model T1-25, cost approx. $20). What follows is a step-by-step description of how to use this calculator to efficiently calculate an exact probability.

This problem was worked on a different set of frequencies than those used in the body of the paper. The frequencies cast in a 2 x 2 contingency table are as follows:

a 6	b 2	a + b = 8
c 3	d 9	c + d = 12
a + c = 9	b + d = 11	N = 20

To calculate the marginal frequencies of the logarithmic equation, [log(a+b)+......] -, the steps are as follows:

a + b	1.	Enter 8
	2.	Press x!
	3.	Press log
	4.	Press SUM
c + d	5.	Enter 12
		Repeat steps 2 thru 4
a + c	6.	Enter 9
		Repeat steps 2 thru 4
b + d	7.	Enter 11
		Repeat steps 2 thru 4
	8.	Press RCL
	9.	Record display <u>26.446776</u>

To calculate N and the cell frequencies - [log(N!) + log(a!).....], the steps are as follows:

	1.	Press Off
	2.	Press ON/C
N	3.	Enter 20
	4.	Press x!
	5.	Press log
	6.	Press SUM
a	7.	Enter 6
		Repeat steps 4 thru 6
b	8.	Enter 2
		Repeat steps 4 thru 6
c	9.	Enter 3
		Repeat steps 4 thru 6
d	10.	Enter 9
		Repeat steps 4 thru 6
	11.	Press RCL
	12.	Record <u>27.882402</u>

To obtain the log of P:

Press ON/C clear display
Enter 26.446776 log(a+b!)......
Press -
Press RCL
Press =
Record <u>-1.4356257</u>

To obtain P:

Press INV
Press log (code for x^{10})
Record .0366754

P = <u>.0367</u>

A similar step-by-step procedure for a Texas Instruments, Model T1-55, for these calculations is available from the author on request. This more advanced calculator has the advantages of capability for programming part of the steps and a sufficient number of memories that the sub-products of the equation can be stored in the machine (i.e., more efficiency and less chance of entering an error).

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